MATH 245 F21, Exam 3 Solutions

1. Carefully define the following terms: symmetric difference, transitive

   Given arbitrary sets R, S, their symmetric difference is the *set* given by $\{x : (x \in R \wedge x \notin S) \vee (x \in S \wedge x \notin R)\}$. Warning: Thm. 8.12 gives two other sets that $R \Delta S$ is equal to, but they are not the definition. Given an arbitrary set S and an arbitrary *relation* R on S, we say that R is transitive if, for every $x, y, z \in S$, $(xRy \wedge yRz) \to xRz$ holds. NOTE: the parentheses are essential, or else the expression is ambiguous. Other common errors include: not giving the categories, not using complete sentences, not using set-builder notation correctly, not giving the quantifiers (for x, y, z) correctly.

2. Carefully state the following theorems: distributivity theorem (for sets), Cantor's theorem

   Given arbitrary sets R, S, T, the distributivity theorem says that both $R \cap (S \cup T) = (R \cap S) \cup (R \cap T)$ and $R \cup (S \cap T) = (R \cup S) \cap (R \cup T)$. Given any set S, Cantor's theorem says that $|S| \neq |2^S|$. NOTE: $|S| < |2^S|$ is not Cantor's theorem, it is a corollary.

3. Let $R = \{x \in \mathbb{Z} : \exists y \in \mathbb{Z}, x = 12y\}$, $S = \{x \in \mathbb{Z} : \exists y \in \mathbb{Z}, x = 5y\}$, $T = \{x \in \mathbb{Z} : \exists y \in \mathbb{Z}, x = 10y\}$. Prove that $R \cap S \subseteq T$.

   Let $x \in R \cap S$ be arbitrary. Then $x \in R \wedge x \in S$, which by simplification (twice) gives $x \in R$ and $x \in S$. Hence, there is some $y \in \mathbb{Z}$ with $x = 12y$ and there is some $t \in \mathbb{Z}$ with $x = 5t$. Note: it is important to use different letters here, not y for both.

   Combining, we get $12y = 5t$, so $5|12y$. Since 5 is prime, either $5|12$ or $5|y$. Since 5 doesn't divide 12, we must have $5|y$. Thus there is some $k \in \mathbb{Z}$ with $5k = y$. Plugging into $x = 12y$, we get $x = 12(5k) = 10(6k)$. Since $6k \in \mathbb{Z}$, we have proved $x \in T$.

4. Let S, T, U be sets with $S \subseteq T \subseteq U$. Prove that $T^c \subseteq S^c$.

   Let $x \in T^c$ be arbitrary. Hence, $x \in U \setminus T$, and hence $x \in U \wedge x \notin T$. By simplification (twice), we get $x \in U$ and $x \notin T$. If $x \in S$, then (since $S \subseteq T$) we would get $x \in T$, which is impossible. Hence $x \notin S$. By conjunction, $x \in U \wedge x \notin S$. Hence $x \in U \setminus S$, and thus $x \in S^c$.

5. Let A, B, C be sets with $A \subseteq B \subseteq C$. Prove that $A \times B \subseteq B \times C$.

   Let $x \in A \times B$ be arbitrary. Hence $x = (u, v)$ with $u \in A$ and $v \in B$. Since $A \subseteq B$, in fact $u \in B$. Since $B \subseteq C$, in fact $v \in C$. Hence $x = (u, v)$ where $u \in B$ and $v \in C$, so $x \in B \times C$.

6. Let $S = \{a, b, c\}$ and $T = 2^S$. Find a relation R from T to S with $|R| = 3$.

   This problem is all about categories and notation. Many answers are possible. A correct answer must be: (1) a set; (2) containing exactly three elements; (3) each of which is an ordered pair; (4) whose first coordinate is a subset of S; and (5) whose second coordinate is an element of S. Here are three correct answers:
   1. $R = \{(\emptyset, a), (\emptyset, b), (\emptyset, c)\}$.
   2. $R = \{(\{a, b\}, a), (\{a\}, a), (\{a, b, c\}, a)\}$.
   3. $R = \{(\{a\}, a), (\{b\}, a), (\{c\}, b)\}$.

7. Prove or disprove: There exists a set S with $|S| \geq 2$, for which the relation $S \times S$ is antisymmetric.

The statement is false. To disprove, we must begin by letting S be an arbitrary set with $|S| \geq 2$. Now, since S contains at least two elements, let a, b (with $a \neq b$) be two of those elements. $S \times S$ contains both (a, b) and (b, a), so it is not antisymmetric.

Note: if S can have 0 or 1 element, the statement becomes true – $S \times S$ is antisymmetric, vacuously. That's not relevant to this question, of course.

8. Prove: For all sets R, S, T, we have $R \cap (S \cup T) \subseteq (R \cap S) \cup (R \cap T)$.
Note: This is (part of) one of our theorems about sets. Don't use that theorem to prove itself.

Let R, S, T be arbitrary sets. Let $x \in R \cap (S \cup T)$ be arbitrary. Then $x \in R \wedge x \in S \cup T$, and therefore $x \in R \wedge (x \in S \vee x \in T)$. We apply the distributive theorem for propositions (a different theorem, from chapter 2) to conclude $(x \in R \wedge x \in S) \vee (x \in R \wedge x \in T)$. Hence $(x \in R \cap S) \vee (x \in R \cap T)$, and finally $x \in (R \cap S) \cup (R \cap T)$.

A common student error (worth 1 point) was neglecting to quantify sets – note that "For all sets R, S, T" appears after "Prove:". Compare with the phrasing "Let R, S, T be sets. Prove:", for which you would not need to quantify the sets (since they were already pre-quantified for you).

9. Prove or disprove: For all sets R, S, T, we have $R \setminus (S \setminus T) = (R \setminus S) \setminus T$.

The statement is false (although $\supseteq$ is true). To disprove, we need explicit choices for sets R, S, T, as well as a careful calculation of all the sets involved. Many solutions are possible. For example, set $R = \{1, 2\}, S = \{1, 3\}, T = \{1, 4\}$. Now, $S \setminus T = \{3\}$ and $R \setminus (S \setminus T) = \{1, 2\}$. On the other hand, $R \setminus S = \{2\}$ and $(R \setminus S) \setminus T = \{2\}$. Lastly, we need to prove the two sets are not equal, for which we need an element that is in one but not the other. We have $1 \in R \setminus (S \setminus T)$ but $1 \notin (R \setminus S) \setminus T$.

10. Find any relation R on $S = \mathbb{Z}$ that simultaneously satisfies both:
(a) R is not symmetric; and (b) $R^{-1} \circ R$ is not reflexive. Be sure to justify your answer.

Many solutions are possible. One of the simplest is $R = \{(1, 2)\}$. Here $R^{-1} = \{(2, 1)\}$ and $R^{-1} \circ R = \{(1, 1)\}$. R is not symmetric because $(1, 2) \in R$ but $(2, 1) \notin R$. $R^{-1} \circ R$ is not reflexive because $(5, 5) \notin R^{-1} \circ R$.

NOTE: For R to not be symmetric you need different integers x, y such that $(x, y) \in R$ but $(y, x) \notin R$. For $R^{-1} \circ R$ to not be reflexive, you need an integer x that does not appear in the first coordinate of R at all. [if $(x, y) \in R$, then $(y, x) \in R^{-1}$ and so $(x, x) \in R^{-1} \circ R$]